Accumulating Calibration Information.

$$\underset{n\times k}{\Psi}\,\underset{k\times m}{\Phi^*} = [\psi_1, \psi_2 \dots \psi_k] \begin{bmatrix} \varphi_1^T \\ \vdots \\ \varphi_k^T \end{bmatrix} \quad n\times m$$

$n = \dim \mathcal{R}$
$m = \dim \mathcal{D}$

$$= \sum_{i=1}^{k} \underbrace{\psi_i \varphi_i^T}_{=G_i} = \sum_{i=1}^{k} G_i = G$$

$$G_i = \psi_i \varphi_i^T$$

$$\Phi \Phi^T = \sum_{i=1}^{k} \underbrace{\varphi_i \varphi_i^T}_{=H_i} = \sum_{i=1}^{k} H_i = \underset{m\times m}{H}$$

$$A_0 = G H^{-1}$$

$$J = \langle H^{-1}, F \rangle S = \operatorname{tr} H^{-1} F \cdot S = \alpha S$$

$$J + S = (\underbrace{\langle H^{-1}, F \rangle}_{=\alpha} + 1) S = (\alpha + 1) S$$

$$\alpha = \operatorname{tr} H^{-1} F = \langle H^{-1}, F \rangle$$

$$\begin{tikzcd}
(\psi_1,\varphi_1) \arrow[r, mapsto] & (G_1,H_1) \arrow[r, Rightarrow] & (G,H) \arrow[d] & & \\
(\psi_2,\varphi_2) \arrow[r, mapsto] & (G_2,H_2) \arrow[r] & \oplus \arrow[r, phantom, "="] & (G+G_2,\ H+H_2) \arrow[r, Rightarrow] & (G,H) \\
 & & \vdots & (G,H) \arrow[dl] & \\
(\psi_k,\varphi_k) \arrow[r] & (G_k,H_k) \arrow[r] & \oplus \arrow[r, Rightarrow] & (G,H). &
\end{tikzcd}$$

$$A_0 = G H^{-1} \qquad J = \underbrace{\operatorname{tr}(H^{-1}F)}_{\alpha} S = \alpha S$$

$$Q = \Big(A_0^* \underbrace{(J+S)}_{=(\alpha+1)S}^{-1} A_0 + F^{-1}\Big)^{-1} =$$

$$= \Big(\frac{1}{\alpha+1} A_0^* S^{-1} A_0 + F^{-1}\Big)^{-1}$$

$$R = Q\, A_0^*(J+S)^{-1} = \frac{1}{\alpha+1} A_0^* S^{-1}$$

$$r = Q F^{-1} x_0$$

$$\hat{x} = Ry + r = Q\Big[\frac{1}{\alpha+1} A_0^* S^{-1} y + F^{-1} x_0\Big]$$

$$E\|\hat{x} - x\|^2 = \operatorname{tr} Q$$

$$\operatorname{Var}(\hat{x} - x) = Q$$

$$\operatorname{Var}(\hat{x}_i - x_i) = Q_{ii}$$

$$\begin{tikzcd}
 & (G,H) \arrow[d] & \\
y \arrow[r] & \boxed{(R,r)} \arrow[r] & \hat{x},\ Q
\end{tikzcd}$$

Repeated measurements of the Same object x

Prior info: $x \sim (x_0, F)$

$$y_1 = Ax + \nu_1$$
$$y_2 = Ax + \nu_2$$
$$\vdots$$
$$y_n = Ax + \nu_n$$

(y_i, A_0, J, S)

n - number of repeated meas
(not to confuse with $n = \dim R$)

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} A \\ \vdots \\ A \end{bmatrix} x + \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_n \end{bmatrix} : \left(\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \begin{bmatrix} A_0 \\ \vdots \\ A_0 \end{bmatrix}, \begin{bmatrix} J & & J \\ & \ddots & \\ J & & J \end{bmatrix}, \begin{bmatrix} S & & 0 \\ & \ddots & \\ 0 & & S \end{bmatrix} \right)$$

$$\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i = Ax + \underbrace{\frac{1}{n} \sum_{i=1}^{n} \nu_i}_{= \bar{\nu}}$$

$$\bar{y} = Ax + \bar{\nu} \qquad \left(\bar{y}, A_0, J, \frac{S}{n} \right)$$

$$\operatorname{Var} \frac{1}{n}\left(\sum \nu_i\right) = \frac{1}{n^2} \operatorname{Var} \sum_{i=1}^{n} \nu_i = \frac{1}{n^2} \sum_{i=1}^{n} S = \frac{n}{n^2} S$$
$$= S/n$$

$$Q = \left(A_0^* \left(J + \frac{1}{n} S\right)^{-1} A_0 + F^{-1}\right)^{-1}$$

$$J = \alpha S$$

$$Q = \left(\left(\alpha + \frac{1}{n}\right)^{-1} A_0^* S^{-1} A_0 + F^{-1}\right)^{-1}$$

$$\hat{x} = Q\left(\left(\alpha + \frac{1}{n}\right)^{-1} A_0^* S^{-1} \bar{y} + F^{-1} x_0\right)$$

$$\alpha = \operatorname{tr} H^{-1} \bar{F} = \operatorname{tr}(\Phi\Phi^*)^{-1} \bar{F}$$

$$E H = E\, \Phi\Phi^T = = E \sum_{i=1}^{k} \varphi_i \varphi_i^T = \sum_{i=1}^{k} \underbrace{E \varphi_i \varphi_i^T}_{\tilde{F}} = k\tilde{F}$$

$$\bar{F} = E x x^T = F + x_0 x_0^T$$

$\Rightarrow \frac{1}{k}\Phi\Phi^T$ is an unbiased est for $\tilde{F}$ – sample second moment of φ_i

$$\frac{1}{k}\Phi\Phi^T \to \tilde{F} \quad \text{as } k \to \infty$$

$$\alpha = \mathrm{tr}(\Phi\Phi^T)^{-1}\bar{F} \sim \frac{1}{k}\underbrace{\mathrm{tr}(\hat{F}^{-1}\bar{F})}_{\mu}$$

$$J = \alpha S = \frac{\mu}{k} S$$ when k is large.

$$Q = \left(\frac{1}{\underbrace{\frac{\mu}{k} + \frac{1}{n}}_{\gamma}} A_0^* S^{-1} A_0 + F^{-1}\right)^{-1}$$

want $\gamma \to +\infty$

$k \to \infty \implies \gamma \to n$

$n \to \infty \implies \gamma \to k/\mu$

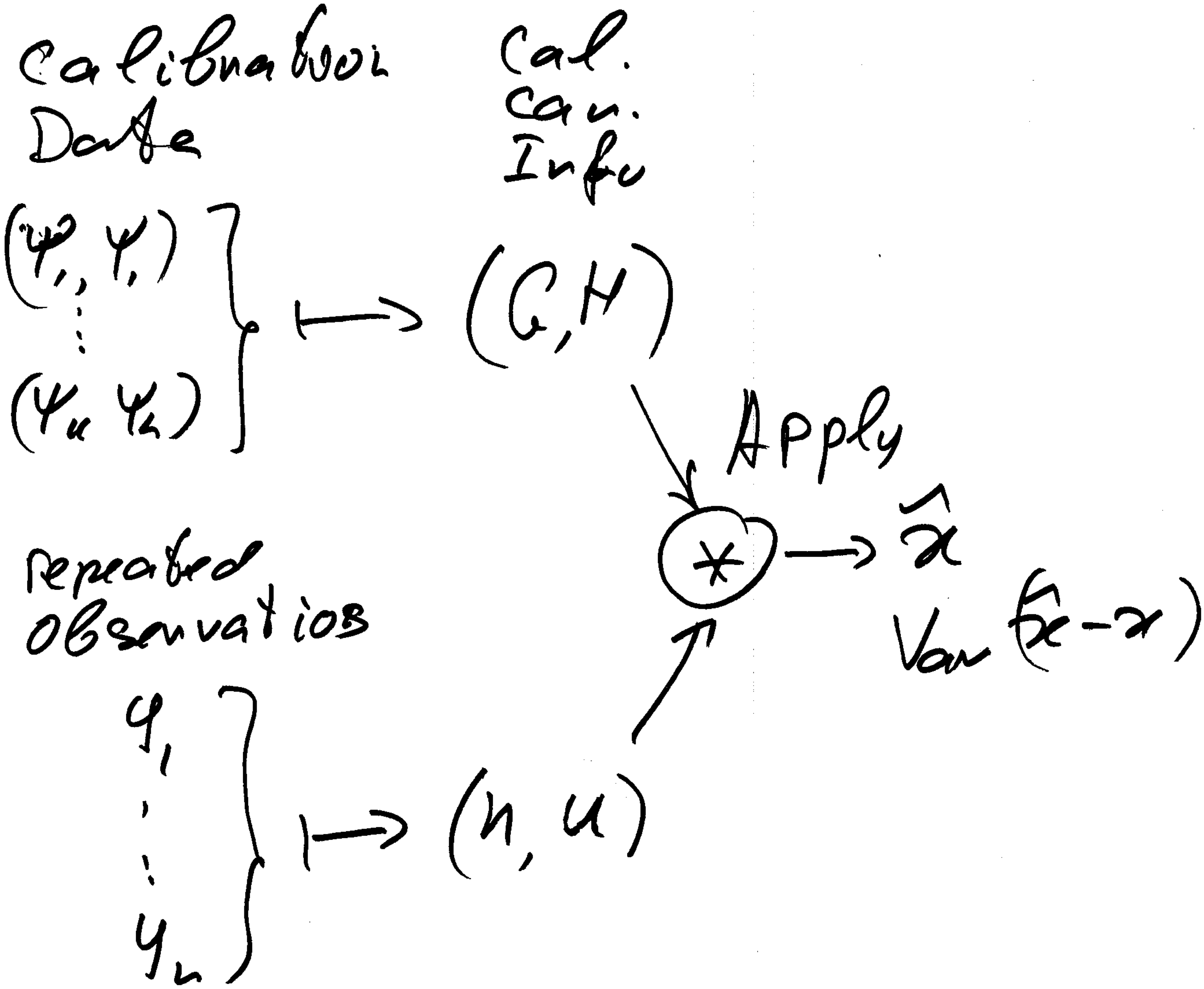
$k, n \to \infty \implies \gamma \to \infty$.

$\implies Q \to 0 \implies \mathrm{Var}(\hat{x} - x) \to 0$

$\mathrm{Var}(\hat{x}_i - x_i) \to 0$

$$\{(\psi_i, \varphi_i)\}_{i=1,\dots k} \to (G, H)$$

$$\{y_i\}_{i=1,\dots n} \to (n, u) \qquad u = \sum_{i=1}^{n} y_i$$



Ballance calibration and repeated measurements

$$Q \approx \left(\gamma A^* S^{-1} A + F^{-1}\right)^{-1}$$

$$\gamma = \frac{1}{\frac{\mu}{k} + \frac{1}{n}}$$

n - num of meas of x

k - num of cal meas

$\mu = tr(\hat{F}^{-1} F)$.

$k, n \to \infty \Rightarrow \gamma \Rightarrow \infty \Rightarrow Q \to C$

Balanced if $\frac{\mu}{k} \approx \frac{1}{n}$ or $\frac{k}{\mu} = n$

$$k = \mu n$$

Suppose that $\hat{F} = F$

$$\mu = tr\big(\underbrace{\hat{F}^{-1} F}_{= I}\big) = tr\, I = m = \dim D$$

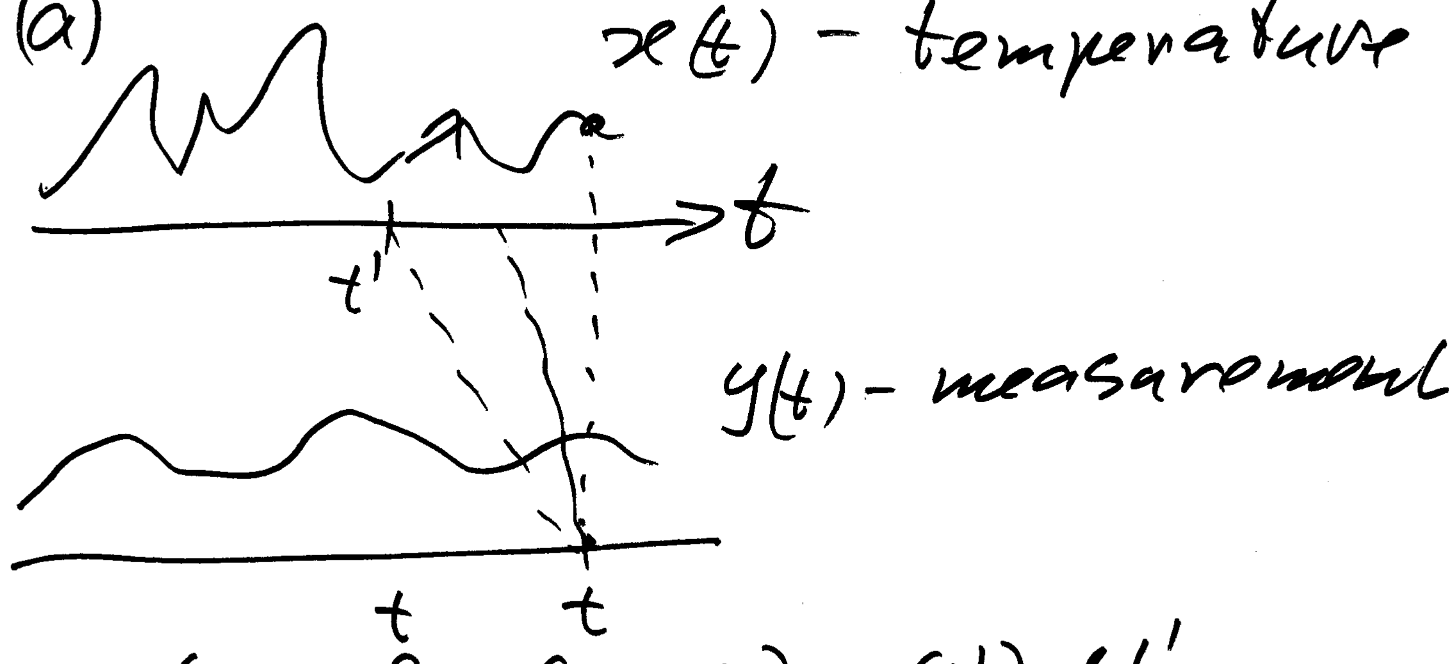
$$\Rightarrow \quad k = m n$$

Num of Calibrations $\approx$ m times larger than num of measurements.

Real-time Processing

* Balance between performance and accuracy
* Process only area of interest
* Split problem into pieces
* process pieces in parallel.
* Combine result from pieces.

(a)



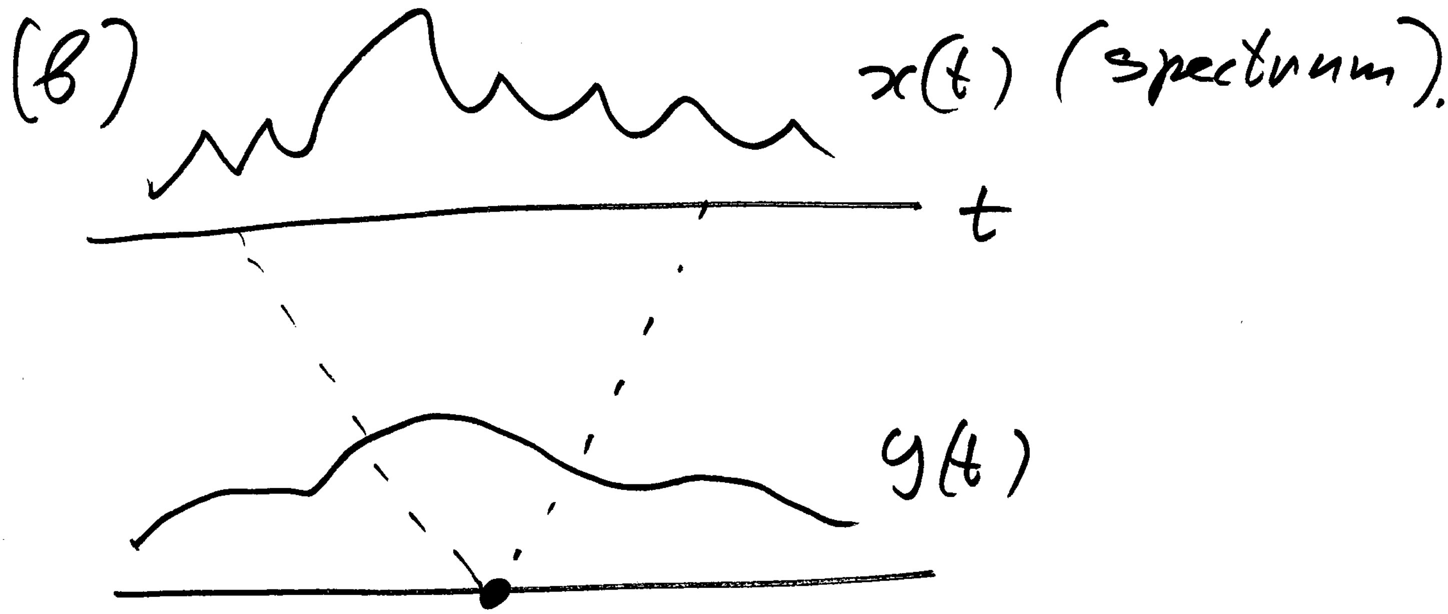
$$y(t) = \int_{-\infty}^{t} a(\underbrace{t-t'}_{\tau})\, x(t')\,dt'$$

$\tau = t - t'$

$t' = t - \tau$

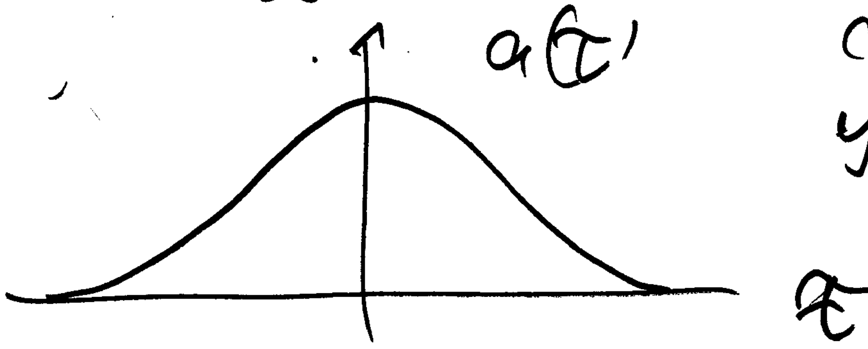
$$= \int_{0}^{\infty} a(\tau)\, x(t-\tau)\,d\tau$$

(b)



$$y(t) = \int_{-\infty}^{\infty} a(t-t')\, x(t')$$

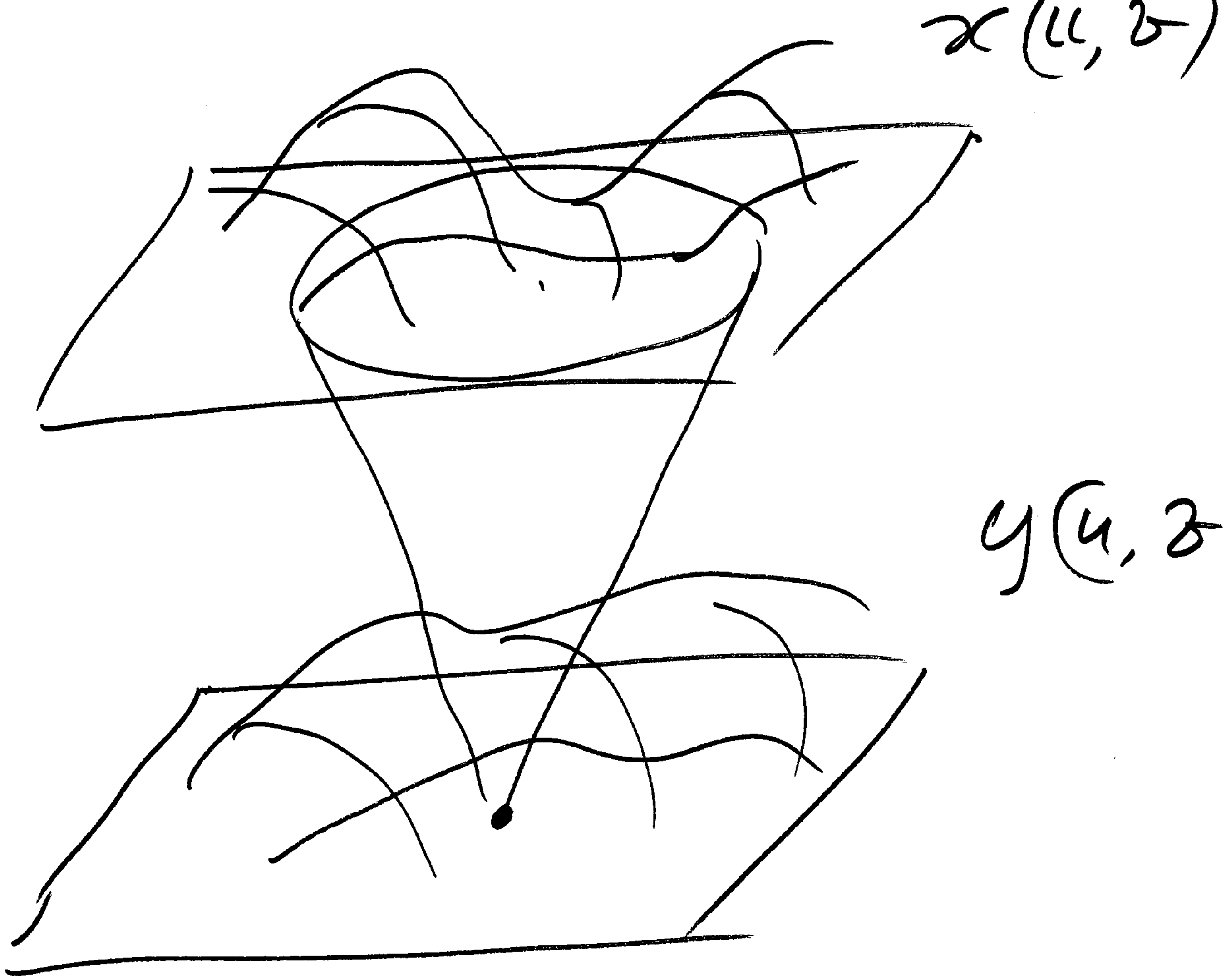
$$= \int_{-\infty}^{\infty} a(\tau)\, x(t-\tau)\, d\tau$$



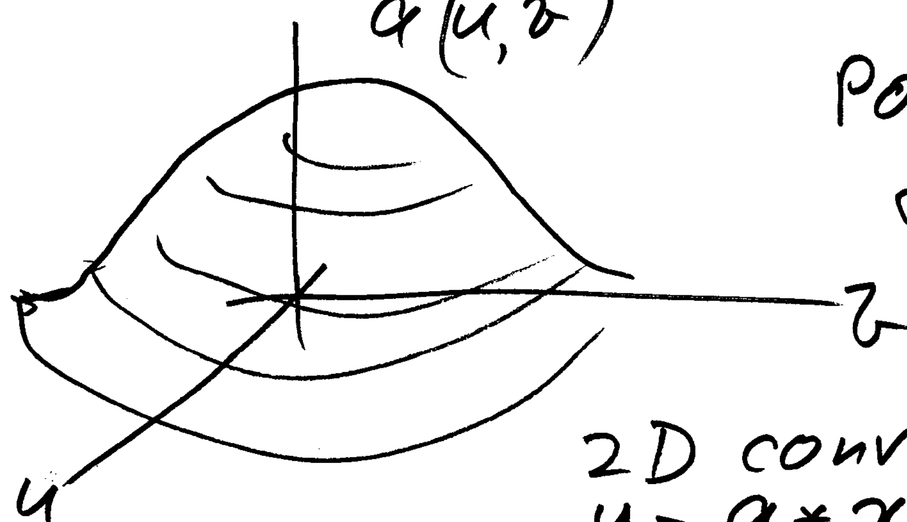
convolution

$y = a * x$

(C) Imaging Systems



$$y(u,v) = \iint_{\mathbb{R}^2} \underline{a(u',v')} \cdot x(u-u', v-v')\, du'dv'$$



Point spread function.

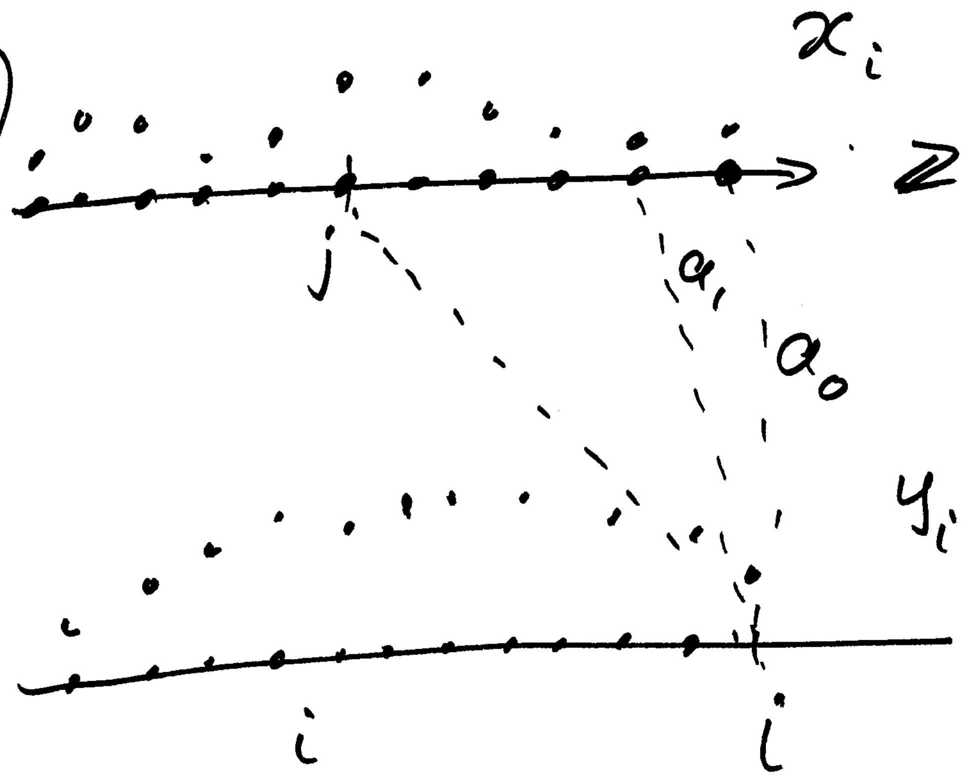
2D convolution
$y = a * x$

Discrete versions

x is a function $\mathbb{Z}$

$x_i \quad i \in \mathbb{Z}$

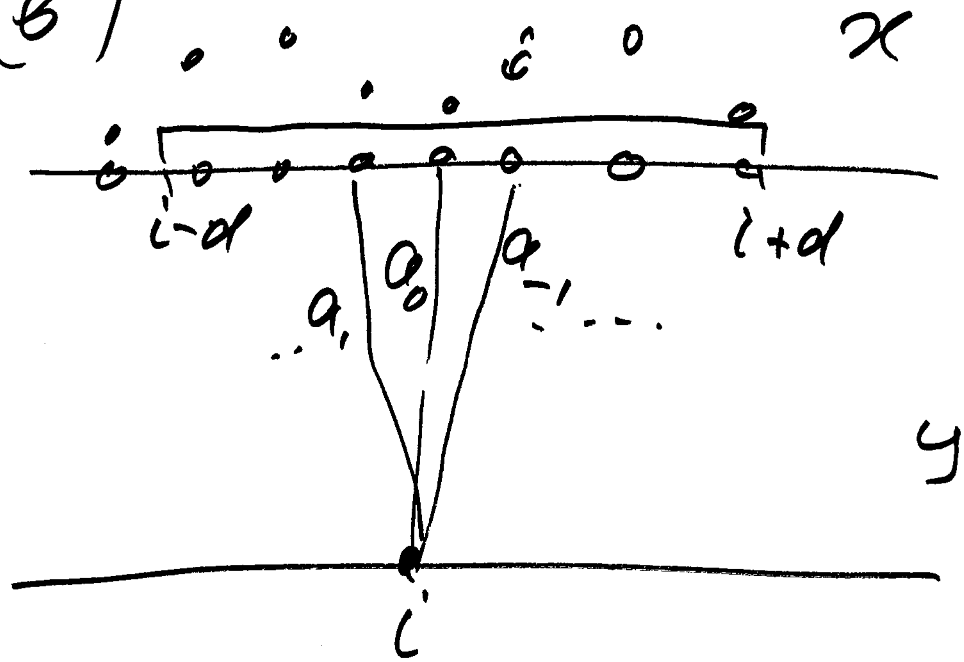
(a)



$$y_i = \sum_{j=-\infty}^{i} a_{\underbrace{i-j}_{=k}} x_j = \qquad k = i-j, \quad j = i-k$$

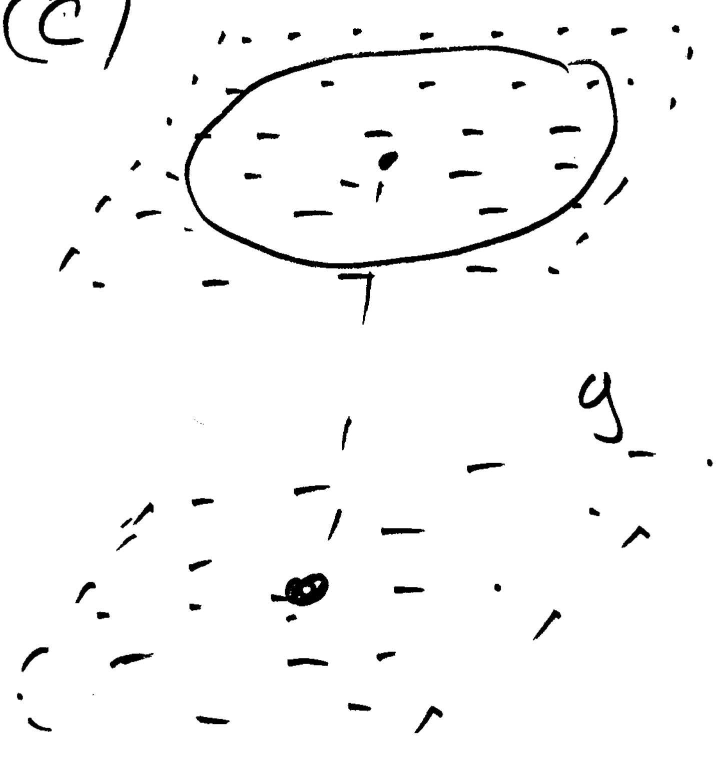
$$= \sum_{k=0}^{+\infty} a_k x_{i-k}$$

(B)



$$y_i = \sum_{j=-d}^{d} a_j x_{i-j}$$

(C)



x - function on $\mathbb{Z}^2$

2D - discrete plane

$$y_{ij} = \sum_{i',j' \in \Delta} a_{i'j'} x_{i-i',j-j'}$$

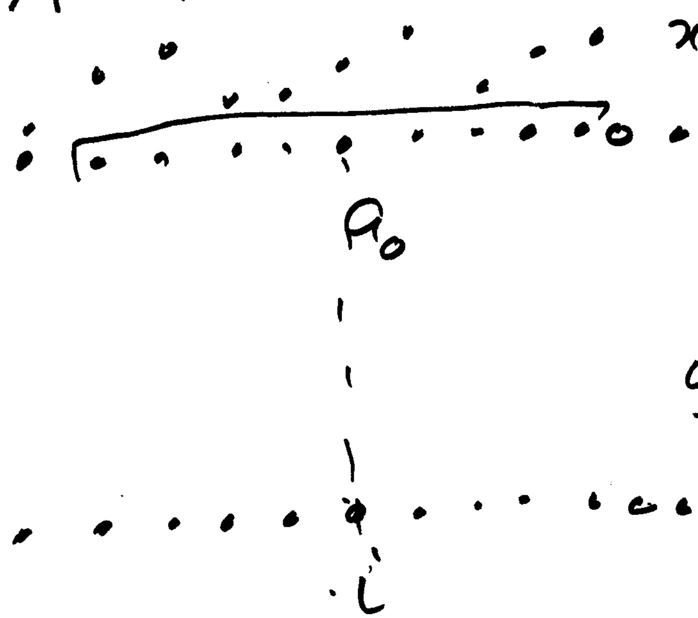
One dimensional discrete field of view.

x – function on $\mathbb{Z}$

$\mathcal{D} = \mathbb{R}^{\mathbb{Z}}$ – set of all functions from $\mathbb{Z}$ to $\mathbb{R}$.

$A : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ $\quad \mathcal{D}, \mathcal{R} = \mathbb{R}^{\mathbb{Z}}$

$x(i) = x_i$



$y = Ax$

$$y_i = \sum_{k \in \Delta_a} a_k x_{i-k}$$

$\Delta_a = [\ell_a, r_a]$ – <u>support</u> of a

if $i \notin \Delta_a \Rightarrow a_i = 0$

$$(a * x)_i = \sum_k a_k x_{i-k}$$

$$y_i = \sum_k a_k x_{i-k} = \sum_j \underbrace{a_{i-j}}_{A_{ij}} x_j$$

$$= \sum_j A_{ij} x_j \qquad A_{ij} = a_{i-j}$$

$$\begin{array}{cc} & j \downarrow \\ i \rightarrow \quad A = & \begin{bmatrix} \ddots & \ddots & & & & & 0 \\ \ddots & a_0 & a_{-1} & \cdots & a_\ell & & \\ & a_1 & a_0 & a_{-1} & \ddots & a_\ell & \\ \ddots & \vdots & a_1 & \ddots & \ddots & \ddots & \ddots \\ & a_r & \ddots & a_1 & a_0 & a_{-1} & a_\ell \\ 0 & & a_r & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \end{array}$$

$$A \longleftrightarrow a$$

lin transformation
(operator)

$BAx \qquad x \xmapsto{A} Ax \xrightarrow{B} BAx$

$A \longleftrightarrow a \qquad \Delta_a = [l_a, r_a]$

$B \longleftrightarrow b \qquad \Delta_b = [l_b, r_b]$

$$B(Ax) = b * (a * x)$$

$$(BAx)_i = \sum_m b_m (a * x)_{i-m} =$$

$$= \sum_m b_m \sum_k a_k \, x_{\underbrace{(i-m)-k}_{i-\underbrace{(m+k)}_{k'}}}$$

$$= \sum_m b_m \sum_{k'} a_{k'-m} \, x_{i-k'} \qquad k' = m+k \quad k = k'-m$$

$$= \sum_{k'} \underbrace{\left[\sum_m b_m \, a_{k'-m} \right]}_{(b*a)_{k'}} x_{i-k'}$$

$$= \big((b*a) * x \big)_i$$

$$b * (a * x) = (b * a) * x$$

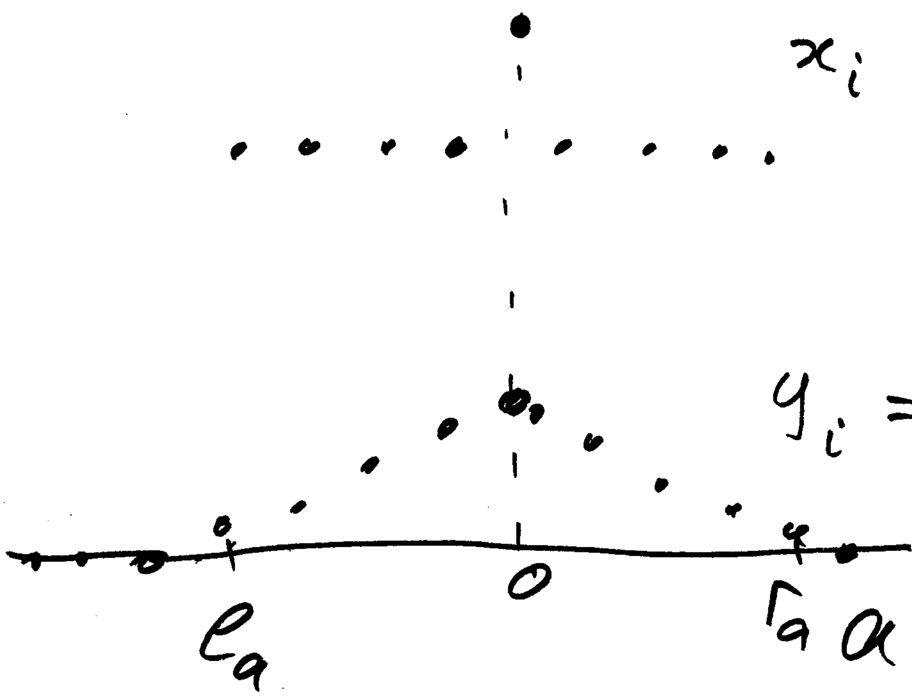
$*$ transitive.

$$BA \longleftrightarrow b * a$$

Another interpretation of a.

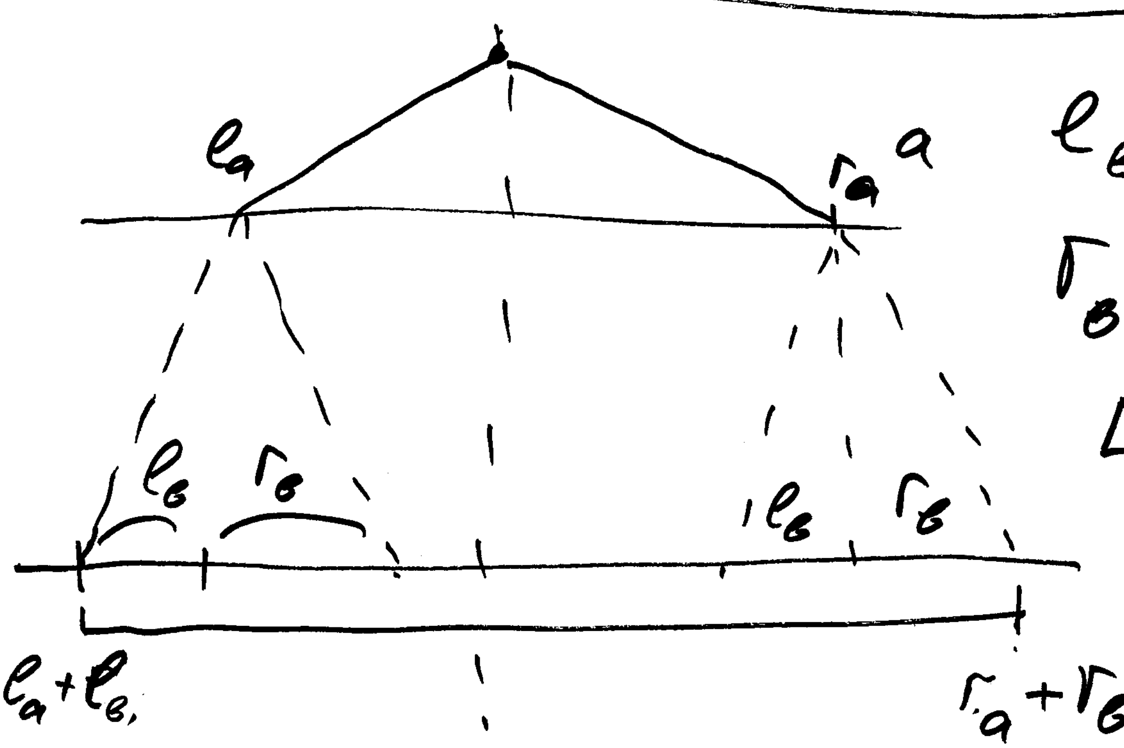
$x = \delta$ discrete version of δ-function

$$x_i = \delta_i = \begin{cases} 1 & i = 0 \\ 0 & i \neq 0 \end{cases}$$



$$y_i = \sum_{k\in\Delta_a} a_k \delta_{i-k} = a_i$$

a - point spread function of A



$$l_{b*a} = l_b + l_a$$

$$r_{b*a} = r_b + r_a$$

$$\Delta_{b*a} = \Delta_b + \Delta_a$$

$A \leftrightarrow a$

$A^T \leftrightarrow a^*$ $\quad a^*_i = a_{-i}$ $\quad \Delta_{a^*} = -\Delta_a$

$\ell_{a^*} = r_a \quad r_{a^*} = \ell_a$

Random vectors on $\mathbb{Z}$

$\nu_i \quad i \in \mathbb{Z}$

ν - random vector in $\mathbb{R}^2$

ν_i - identically distributed

$E\nu_i = 0 \qquad E\nu_i^2 = \sigma^2$

$cov(\nu_i, \nu_j) = E\nu_i\nu_j = S_{ij}$

Assume that

$S_{i+n,\, j+n} = S_{ij} \Rightarrow$

$S_{ij} = S_{i-j,\,0} = s_{i-j}$

covariance function of ν

Homogeneous random field

(stationary stochastic process when i is time).

$$\mu = A\nu = a * \nu$$

$$E\mu_i = E\sum_K a_k \nu_{i-k} = \sum_k a_k \underbrace{E\nu_{i-k}}_{=0} = 0$$

$$E\mu_i\mu_j = E\left[\sum_m a_m \nu_{i-m} \cdot \sum_K a_k \nu_{j-k}\right]$$

$$= \sum_{m,k} a_m \underbrace{\left[E\,\nu_{i-m} \cdot \nu_{j-k}\right]}_{S_{\underbrace{i-j}_{n}+k-m}} a_k$$

$$= \sum_{m,k} a_m \, S_{n+k-m} \, a_k$$

$$(a * s * a^*)_n = \sum_e (a*s)_e \cdot a^*_{n-e}$$

$$= \sum_{e,m} a_m s_{e-m} \cdot a_{\underbrace{e-n}_{=k}}$$

$k = e - n$

$e = k + n$

$$= \sum_{m,k} a_n \, S_{k+n-m} \, a_k$$

$$\operatorname{cov}\mu = a * S * a^*$$

if ν_i - indep $\Rightarrow S = \sigma^2\delta \Rightarrow \operatorname{cov}\mu = \sigma^2 a * a^*$