Examples. (cont.)

(c) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad S = \sigma^2 I$

$\hat{x} = \frac{1}{2}\begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \end{bmatrix}$ $\quad \hat{x}_1 = \frac{y_1 + y_2}{2}$

$\hat{x}_2 = \frac{y_1 - y_2}{2}$

$\mathrm{Var}(\hat{x}_1) = \mathrm{Var}(\hat{x}_2) = \frac{\sigma^2}{2} < \underbrace{\frac{2}{3}\sigma^2}_{\text{case}(6)} < \underbrace{\sigma^2}_{\text{case}(9)}$

(d) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ $\quad S = \sigma^2 \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}$

$\mathrm{Var}(\hat{x}) = (A^* S^{-1} A)^{-1}$

$\hat{x} = Ry = \underbrace{(A^* S^{-1} A)^{-1} A^* S^{-1}}_{R} y$

$A^* S^{-1} = S^{-1}$

$\mathrm{Var}\, \hat{x} = (I S^{-1} I)^{-1} = S$

$R = S \cdot I S^{-1} = I$

Unbiased est $\Leftrightarrow RA = I$

Since $A = I \Rightarrow R = I$.

$\hat{x} = y \quad \hat{x}_1 = y_1 \quad \hat{x}_2 = y_2$

$\mathrm{Var}\, \hat{x}_1 = \sigma^2 = \mathrm{Var}\, \hat{x}_2$

(e) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $S = \sigma^2 \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}$

$S^{-1} = \sigma^{-2} \frac{1}{1-r^2} \begin{bmatrix} 1 & -r \\ -r & 1 \end{bmatrix}$ $0 \le r \le 1$

$$A^* S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sigma^2(1-r^2)} \begin{bmatrix} 1 & -r \\ -r & 1 \end{bmatrix}$$

$$= \frac{1}{\sigma^2(1-r^2)} \begin{bmatrix} 1-r & 1-r \\ 1+r & -1-r \end{bmatrix}$$

$$\mathrm{Var}(\hat{x}) = (A^* S^{-1} A)^{-1}$$

$$= \sigma^2(1-r^2) \left(\begin{bmatrix} 1-r & 1-r \\ 1+r & -1-r \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^{-1}$$

$$= \sigma^2(1-r^2) \begin{bmatrix} 2(1-r) & 0 \\ 0 & 2(1+r) \end{bmatrix}^{-1}$$

$$= \frac{\sigma^2(1-r^2)}{2} \begin{bmatrix} \frac{1}{1-r} & 0 \\ 0 & \frac{1}{1+r} \end{bmatrix}$$

$$= \frac{\sigma^2}{2} \begin{bmatrix} 1+r & 0 \\ 0 & 1-r \end{bmatrix}$$

$$\mathrm{Var}(\hat{x}_1) = \frac{\sigma^2}{2}(1+r) \geq \frac{\sigma^2}{2}$$ case(c) worse.

$$\mathrm{var}(\hat{x}_2) = \frac{\sigma^2}{2}(1-r) \leq \frac{\sigma^2}{2}$$ — " — better

$$R = (A^* S^{-1} A)^{-1} A^* S^{-1} =$$

$$= \frac{\sigma^2}{2}\begin{bmatrix} 1+r & 0 \\ 0 & 1-r \end{bmatrix} \frac{1}{\sigma^2(1-r^2)} \begin{bmatrix} 1-r & 1-r \\ 1+r & -1-r \end{bmatrix}$$

$$= \frac{1}{2(1-r^2)} \begin{bmatrix} 1-r^2 & 1-r^2 \\ 1-r^2 & -(1-r^2) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = R$$ for case(c)

$$R = \underbrace{(A^* S^{-1} A)^{-1}}_{A^{-1} S A^{*-1}} A^* S^{-1} = \underbrace{A^{-1} S \underbrace{A^{*-1} A^*}_{=I} S^{-1}}_{=1}$$

$$= A^{-1}$$

$RA = I \Rightarrow$ since A is invertible $R = A^{-1}$

$$\hat{x} = Ry = \begin{bmatrix} \dfrac{y_1 + y_2}{2} \\ \dfrac{y_1 - y_2}{2} \end{bmatrix}$$ as in (c).

$$y_1 = x_1 + x_2 + \underbrace{\varepsilon_1 + \varepsilon_0}_{\nu_1}$$

$$\mathrm{Var}(\varepsilon_1) = \mathrm{Var}(\varepsilon_2) = \sigma_1^2$$

$$y_2 = x_1 - x_2 + \underbrace{\varepsilon_2 + \varepsilon_0}_{\nu_2}$$

$$\mathrm{Var}\ \varepsilon_0 = \sigma_0^2$$

$$r = \frac{\sigma_0^2}{\sigma_1^2 + \sigma_0^2}$$

$$\sigma^2 = \sigma_1^2 + \sigma_0^2 = \text{const}$$

$r = 0 \Rightarrow$

$$\sigma_0^2 = 0$$

$$\sigma_1^2 = \sigma^2 \quad (\text{uncorr.})$$

$r = 1 \quad \sigma_0^2 = \sigma^2 \quad \sigma_1^2 = 0$

$$y_1 = x_1 + x_2 + \varepsilon_0$$

$$y_2 = x_1 - x_2 + \varepsilon_0$$

$$\Rightarrow y_1 - y_2 = 2x_2 \Rightarrow \hat{x}_2 = \frac{y_1 - y_2}{2} = x_2$$

$\mathrm{Var}(\hat{x}_2) = 0.$ Precise!

$$y_1 + y_2 = 2x_1 + 2\varepsilon_0$$

best est $\hat{x}_1 = \dfrac{y_1 + y_2}{2} = x_1 + \varepsilon_0$

$$\Rightarrow \mathrm{Var}(\hat{x}_1) = \mathrm{Var}(\varepsilon_0) = \sigma^2.$$

Linear regression as a particular case of linear estimation.

$$y_i = f_1(x_i)a_1 + \cdots + f_m(x_i)a_m + \varepsilon_i$$

$i = 1, \ldots n$

$$y_i = F(x_i)a + \varepsilon_i \qquad a = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$$

unknown.

$$\underline{y = Ba + \varepsilon}$$

$$B = \begin{bmatrix} F(x_1) \\ \vdots \\ F(x_n) \end{bmatrix}$$

Optimal linear estimation

$$E\|Ry - a\|^2 \sim \min_R \; : \; RB = I$$

$$\hat{a} = Ry = \left(B^T \cdot \sigma^{-2} I \cdot B\right)^{-1} B^T \cdot \sigma^{-2} I \cdot y$$

$$= \left(B^T B\right)^{-1} B^T y$$

$$\mathrm{Var}(\hat{a}) = \left(B^T B\right)^{-1}$$

In lin regression

$$\|y - Ba\|^2 \sim \min_a \Rightarrow \hat{a}$$

Combining linear experiments.

$$y = Ax + \nu$$

$$z = Bx + \mu$$

x - unknown $x \in D$

$y \in R$ $z \in Q$

$\nu \in R$, $\mu \in Q$ ν, μ - indep.

$\nu \sim (0, S)$, $\mu \sim (0, T)$

$S = Var(\nu)$ $T = Var\,\mu$

$S : R \to R$ $T : Q \to Q$

Raw information.

$$\begin{array}{l}(y, A, S) \\ (z, B, T)\end{array} \oplus ?$$

$\mathcal{R}$ and $\mathcal{Q}$ – vector spaces

$\begin{bmatrix} y \\ z \end{bmatrix} \in \mathcal{R} \times \mathcal{Q}$ product of vector spaces $\mathcal{R}$ and $\mathcal{Q}$

$y \in \mathcal{R}$, $z \in \mathcal{Q}$

"+" : $$\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} \underset{\mathcal{R}\times\mathcal{Q}}{+} \begin{bmatrix} y_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 \underset{\mathcal{R}}{+} y_2 \\ z_1 \underset{\mathcal{Q}}{+} z_2 \end{bmatrix}$$

"$\alpha \cdot$" $$\alpha \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \alpha y \\ \alpha z \end{bmatrix}$$

Inner product:

$$\left\langle \begin{bmatrix} y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} y_2 \\ z_2 \end{bmatrix} \right\rangle_{\mathcal{R}\times\mathcal{Q}} = \langle y_1, y_2 \rangle_{\mathcal{R}} + \langle z_1, z_2 \rangle_{\mathcal{Q}}$$

$$\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = \sqrt{\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle} = \sqrt{\langle x, x \rangle + \langle y, y \rangle}$$

$$= \sqrt{\|x\|^2 + \|y\|^2}$$

Let $\mathcal{R} = \mathbb{R}^n \qquad \mathcal{Q} = \mathbb{R}^m$

$$\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n \qquad \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} \in \mathbb{R}^m$$

$$n+m\left\{\begin{array}{r} n\left\{\vphantom{\begin{matrix} y_1 \\ \vdots \\ y_n \end{matrix}}\right. \\ m\left\{\vphantom{\begin{matrix} z_1 \\ \vdots \\ z_m \end{matrix}}\right. \end{array}\right. \begin{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \\ \begin{bmatrix} z_1 \\ \vdots \\ z_m \end{bmatrix} \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \\ z_1 \\ \vdots \\ z_m \end{bmatrix} \in \mathbb{R}^{n+m}$$

Let $A : \mathcal{D} \to \mathcal{R}$

$\qquad B : \mathcal{D} \to \mathcal{Q}$

$$\begin{bmatrix} A \\ B \end{bmatrix} : \mathcal{D} \to \mathcal{R} \times \mathcal{Q}$$

$$\forall x \in \mathcal{D} \quad \begin{bmatrix} A \\ B \end{bmatrix} x = \begin{bmatrix} Ax \\ Bx \end{bmatrix} \in \mathcal{R} \times \mathcal{Q}$$

If A, B - linear $\Rightarrow \begin{bmatrix} A \\ B \end{bmatrix}$ - linear.

If $A : \mathcal{D} \to \mathcal{R}$
$B : \mathcal{E} \to \mathcal{R}$

$$[A \; B] : \mathcal{D} \times \mathcal{E} \to \mathcal{R}$$

$$\forall \begin{bmatrix} y \\ z \end{bmatrix} \in \mathcal{D} \times \mathcal{E}$$

$$[A \; B] \begin{bmatrix} y \\ z \end{bmatrix} = Ay + Bz \in \mathcal{R}$$

$A : \mathcal{D} \to \mathcal{R}$

$B : \mathcal{E} \to \mathcal{R}$

$C : \mathcal{D} \to \mathcal{Q}$

$D : \mathcal{E} \to \mathcal{Q}$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{D} \times \mathcal{E} \to \mathcal{R} \times \mathcal{Q}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} Ay + Bz \\ Cy + Dz \end{bmatrix}$$

Let $\nu \in \mathcal{R}$ and $\mu \in Q$
be random vectors

$\begin{bmatrix} \nu \\ \mu \end{bmatrix} \in \mathcal{R} \times Q$ – random vector

$$E \begin{bmatrix} \nu \\ \mu \end{bmatrix} = \begin{bmatrix} E\nu \\ E\mu \end{bmatrix}$$

We will assume that $E\nu = 0$
$E\mu = 0$

$\mathrm{Var}(\nu) = S$

$\mathrm{Var}(\mu) = T$

$\mathrm{Var}\left(\begin{bmatrix} \nu \\ \mu \end{bmatrix}\right) : \mathcal{R} \times Q \to \mathcal{R} \times Q$

$$\mathrm{Var}\left(\begin{bmatrix} \nu \\ \mu \end{bmatrix}\right) \begin{bmatrix} y \\ z \end{bmatrix} = E \left\langle \begin{bmatrix} \nu \\ \mu \end{bmatrix}, \begin{bmatrix} y \\ z \end{bmatrix} \right\rangle \begin{bmatrix} \nu \\ \mu \end{bmatrix}$$

$$= E \left(\langle \nu, y \rangle + \langle \mu, z \rangle \right) \begin{bmatrix} \nu \\ \mu \end{bmatrix}$$

$$= \begin{bmatrix} E\langle \nu, y \rangle \nu + E\langle \mu, z \rangle \nu \\ E\langle \nu, y \rangle \mu + E\langle \mu, z \rangle \mu \end{bmatrix}$$

$$= \begin{bmatrix} Sy & \langle \overbrace{E\mu}^{0}, z \rangle \overbrace{E\nu}^{0} \\ 0 & Tz \end{bmatrix}$$

$$= \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$$

If $Var(\nu) = S$, $Var(\mu) = T$, ν and μ indep $\Rightarrow$

$$Var\begin{bmatrix}\nu\\ \mu\end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}$$

Combination of measurements.

$$\begin{array}{l} y = Ax + \nu \\ z = Bx + \mu \end{array} \Rightarrow \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} x + \begin{bmatrix} \nu \\ \mu \end{bmatrix}$$

combination in Raw form

$$\left.\begin{array}{l} (y, A, S) \\ (z, B, T) \end{array}\right. \oplus = \left(\begin{bmatrix} y \\ z \end{bmatrix}, \begin{bmatrix} A \\ B \end{bmatrix}, \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix} \right)$$

For n measurements

$$\left.\begin{bmatrix} (y_1, A_1, S_1) \\ \vdots \\ (y_n, A_n, S_n) \end{bmatrix}\right. \oplus = \left(\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}, \begin{bmatrix} S_1 & & 0 \\ & \ddots & \\ 0 & & S_n \end{bmatrix} \right)$$

$$y_1 = A_1 x + v_1 \qquad v_1 \sim (0, S_1)$$
$$y_2 = A_2 x + v_2 \qquad v_2 \sim (0, S_2)$$
$$y_1, v_1 \in \mathcal{R}_1 \quad y_2, v_2 \in \mathcal{R}_2$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \quad S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$$

$$A : \mathcal{D} \to \mathcal{R}_1 \times \mathcal{R}_2 \qquad y \in \mathcal{R}_1 \times \mathcal{R}_2$$

$$\hat{x} = Ry = (A^* S^{-1} A)^{-1} A^* S^{-1} y$$

$$\mathrm{Var}(\hat{x}) = (A^* S^{-1} A)^{-1}$$

$$A^* S^{-1} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^* \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} A_1^* & A_2^* \end{bmatrix} \begin{bmatrix} S_1^{-1} & 0 \\ 0 & S_2^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} A_1^* S_1^{-1} + A_2^* \cdot 0 & A_1^* 0 + A_2^* S_2^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} A_1^* S_1^{-1} & A_2^* S_2^{-1} \end{bmatrix} : \mathcal{R}_1 \times \mathcal{R}_2 \to \mathcal{D}.$$

$$A^* S^{-1} A = \begin{bmatrix} A_1^* S_1^{-1} & A_2^* S_2^{-1} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

$$= \underbrace{A_1^* S_1^{-1} A_1}_{T_1} + \underbrace{A_2^* S_2^{-1} A_2}_{T_2} : \mathcal{D} \to \mathcal{D}$$

$$= T_1 + T_2 = T$$

$$A^* S^{-1} y = \begin{bmatrix} A_1^* S_1^{-1} & A_2^* S_2^{-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \underbrace{A_1^* S_1^{-1} y_1}_{= b_1} + \underbrace{A_2^* S_2^{-1} y_2}_{= b_2}$$

$$= b_1 + b_2 = b \in \mathcal{D}.$$

$$T_i = A_i^* S_i^{-1} A, \quad b_i = A_i^* S_i^{-1} y,$$

$$(y_1, A_1, S_1) \mapsto (T_1, b_1)$$

$$\oplus = (T_1 + T_2,\ b_1 + b_2)$$

$$(y_2, A_2, S_2) \mapsto (T_2, b_2)$$

$$T \in \mathcal{D} \to \mathcal{D} \qquad b \in \mathcal{D}$$

$$\hat{x} = T^{-1} b \qquad \mathrm{var}(\hat{x}) = T^{-1}$$

n measurements:

$$\left.\begin{array}{l}(y_1, A_1, S_1) \mapsto (T_1, \tau_1) \\ \vdots \\ (y_n, A_n, S_n) \mapsto (T_n, \tau_n)\end{array}\right] \oplus =$$

$$= \left(\sum_{i=1}^{n} T_i, \sum_{i=1}^{n} \tau_i\right) = (T, \tau)$$

(T_i, τ_i), (T, τ) - <u>canon. info</u>

Assume that S_i - invertible.

T_i not necessarily invertible.
(not enough for estimation)

Examples. $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$

(1) $y_1 = \underbrace{x_1} + \nu_1 \qquad y_1 \in \mathbb{R}^1$

$A_1 = [1 \;\; 0] \qquad A_1 x = [1 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1$

$S_1 = \sigma^2 = \text{var}\, \nu_1$

Raw form: $(y_1, A_1, S_1) = (y, [1 \; 0], \sigma^2)$

Can Info. $T_1 = A_1^T S_1^{-1} A_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sigma^{-2} [1 \;\; 0]$

$= \sigma^{-2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$z_1 = A_1^T S_1^{-1} y_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sigma^{-2} y_1 = \sigma^{-2} \begin{bmatrix} y_1 \\ 0 \end{bmatrix}$

$(T_1, z_1) = \left(\begin{bmatrix} \sigma^{-2} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \sigma^{-2} y_1 \\ 0 \end{bmatrix} \right)$

T_1 - not inventible

can not yet est $\hat{x}$, and $\text{Var}(\hat{x})$

(2) $y_2 = x_2 + \nu_2 \qquad y_2 \in \mathbb{R}^1$

$A_2 = [0, 1] \qquad S_2 = \sigma^2$

$T_2 = \begin{bmatrix} 0 & 0 \\ 0 & \sigma^{-2} \end{bmatrix} \qquad z_2 = \begin{bmatrix} 0 \\ \sigma^{-2} y_2 \end{bmatrix}$

T_2 - not invert.

(3) $y_3 = x_1 + x_2 + v_3 \qquad y_3 \in \mathbb{R}$

$$A_3 = [1 \quad 1] \qquad S_3 = \sigma^2$$

$$T_3 = A_3^T S_3^{-1} A_3 = \begin{bmatrix}1\\1\end{bmatrix} \sigma^{-2} [1 \quad 1] = \sigma^{-2} \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

Singular

$$z_3 = A_3^T S_3^{-1} y_3 = \sigma^{-2} \begin{bmatrix}1\\1\end{bmatrix} y_3 = \sigma^{-2} \begin{bmatrix}y_3\\y_3\end{bmatrix}$$

Combine (1)+(2)+(3)

$$\bigoplus_{i=1}^{3} (T_i, z_i) = \left(\sigma^{-2} \left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \right), \sigma^{-2} \left(\begin{bmatrix}y_1\\0\end{bmatrix} + \begin{bmatrix}0\\y_2\end{bmatrix} + \begin{bmatrix}y_3\\y_3\end{bmatrix} \right) \right)$$

$$= \left(\sigma^{-2} \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}, \sigma^{-2} \begin{bmatrix}y_1 + y_3\\y_2 + y_3\end{bmatrix} \right)$$

$$= (T, z)$$

$$\mathrm{Var}(\hat{x}) = T^{-1} = \left(\sigma^{-2} \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix} \right)^{-1}$$

$$= \sigma^2 \frac{1}{3} \begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}$$

$$\hat{x} = T^{-1} z = \sigma^2 \frac{1}{3} \begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix} \sigma^{-2} \begin{bmatrix}y_1 + y_3\\y_2 + y_3\end{bmatrix}$$

(B) = (1)+(2)+(3)
$$= \frac{1}{3} \begin{bmatrix}2(y_1 + y_3) - (y_2 + y_3)\\-(y_1 + y_3) + 2(y_2 + y_3)\end{bmatrix} = \frac{1}{3} \begin{bmatrix}2y_1 - y_2 + y_3\\-y_1 + 2y_2 + y_3\end{bmatrix}$$

(4) $y_4 = x_1 - x_2 + v_4$

$$A_4 = \begin{bmatrix} 1 & -1 \end{bmatrix} \qquad S_4 = \sigma^2$$

$$T_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sigma^{-2} \begin{bmatrix} 1 & -1 \end{bmatrix} = \sigma^{-2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$z_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \sigma^{-2} y_4 = \sigma^{-2} \begin{bmatrix} y_4 \\ -y_4 \end{bmatrix}$$

$$\bigoplus_{i=1}^{4} (T_i, z_i) = \bigoplus_{i=1}^{3} (T_i, z_i) \oplus (T_4, z_4)$$

$$= \left(\sigma^{-2} \left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right), \sigma^{-2} \left(\begin{bmatrix} y_1 + y_3 \\ y_2 + y_3 \end{bmatrix} + \begin{bmatrix} y_4 \\ -y_4 \end{bmatrix} \right) \right)$$

$$= \left(\sigma^{-2} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \sigma^{-2} \begin{bmatrix} y_1 + y_3 + y_4 \\ y_2 + y_3 - y_4 \end{bmatrix} \right)$$

$$\mathrm{Var}\, \hat{x} = \left(\sigma^{-2} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right)^{-1} = \sigma^2 \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$\mathrm{var}\, \hat{x}_1 = \mathrm{Var}\, \hat{x}_2 = \frac{\sigma^2}{3}.$$